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LETTER TO THE EDITOR

Existence of stable periodic orbits in the x^2y^2 potential: a semiclassical approach

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Abstract. We use the semiclassical periodic orbit theory to identify the recently discovered one-parameter family of stable periodic orbits in the x^2y^2 potential occupying an area of 0.005% on the surface of section. We also indicate the presence of another stable family of periodic orbits of higher length. The sensitivity of the method provides hope for ruling out ergodicity in other systems.

Generic Hamiltonian systems are known to possess a rich dynamical structure [1]. While some orbits in phase space reside on invariant tori, others explore the entire constant energy surface chaotically. The two limits spanned by this generic category are the integrable and the chaotic systems (strongly ergodic). For N-dimensional systems, the former case is characterized by periodic orbits that occur in $(N-1)$-parameter families while in case of the latter, these are isolated and unstable. Examples of systems known to be fully chaotic are the Sinai [2] and Bunumovich [3] billiards, and the motion on a Riemann surface of constant negative curvature [4]. Amongst the analytic Hamiltonians, a likely candidate has been the motion in the potential [5, 6] $V(x, y) = x^2y^2/2$. The importance of this system in Yang-Mills theory [5] has been a motivating factor for its study and a number of researchers have in various ways argued that the system is strongly ergodic, with all periodic orbits isolated and unstable (see [5] for details). The recent work of Dahlqvist and Russberg [7], however, indicates otherwise. There does exist at least one family of stable periodic orbits in the system.

The discovery, though not quite accidental, does raise an important question. Is it possible to extract details of the nature of periodic orbits through other more practical means? The well established connection between the periodic orbits and the semiclassical density of energy eigenstates does provide an answer. In the following we report our first results for an analytic Hamiltonian system and show that starting from the quantum spectra, it is possible to 'read off' the lengths and (more important) the nature of periodic orbits of the underlying classical system. We shall, at the moment, restrict ourselves to the x^2y^2 potential and concentrate on the one-parameter family of periodic orbits discovered recently by Dahlqvist and Russberg [7]. We also indicate the presence of another stable family of periodic orbits at higher length which probably occupies a larger area in phase space.

The Hamiltonian $H = p_x^2/2 + p_y^2/2 + x^2y^2/2$ exhibits dynamical similarity due to the homogeneity of both the kinetic and potential energies. As a result, all periodic orbits depend trivially on the energy. The scaling relation between the orbits at different energies are thus simple. A trajectory $(p(t), q(t))$ at energy E is determined by a trajectory $(p_0(t), q_0(t))$ by the scaling $p(\tau)=(E/E_0)^{1/2}p_0(t)$, $q(\tau)=(E/E_0)^{1/4}q_0(t)$ and the scaled time $\tau=(E/E_0)^{-1/4}$.

The classical action $S=\oint \boldsymbol{p}\,\mathrm{d}\boldsymbol{q}$ thus satisfies the scaling $S(E)=S(E_0)(E/E_0)^{3/4}$. For a periodic orbit $S=\int \boldsymbol{p}\,\mathrm{d}\boldsymbol{q}=(E)^{3/4}f(l_\gamma)$, where $f(l_\gamma)$ is a constant depending only on the periodic orbit. It is actually the scaled action of the orbit.

For the family of stable periodic orbits recently discovered the quantity $f(l_\gamma)=34.8$. It was computed using a fourth-order Runge-Kutta method and numerical errors are under control. The orbit is stable [7] and moreover the total area of the elliptic region in the surface of section was found to be 0.005%. Our efforts in the following would be to determine the scaled action, $f(l_\gamma)$, of this stable family of periodic orbits using only the quantum spectra.

It is now well established that the semiclassical spectrum of time independent Hamiltonian systems is determined by the periodic orbits of the underlying dynamics. The main result of the periodic orbit (PO) theory, developed by Gutzwiller, Balian and Bloch and Berry and Tabor (for a review see Berry [8]) is that the semiclassical level density, $d(E)=\sum_i \delta(E-E_i)$, can be written as

$$\begin{aligned} d(E)&=\langle d(E)\rangle+d_{\mathrm{osc}}(E)\\ &=\langle d(E)\rangle+\sum_j\sum_\gamma A_{\gamma,j}(E)\cos\{j[S_\gamma(E)-\alpha_\gamma]\} \end{aligned} \tag{1}$$

where γ labels all the primitive periodic orbits of the classical system with action $S_\gamma(E)$ and phase α_γ. The sum describes oscillatory correction to the mean level density, $\langle d(E)\rangle$, which is given by the size of the energy shell in phase space. The amplitudes $A_{\gamma,j}$ of the oscillation depend on two aspects of the periodic orbits—whether they are stable and whether they are isolated. If the orbit is exponentially unstable (chaotic), $A_{\gamma,j}$ decreases exponentially with j whereas $A_{\gamma,j}$ oscillates with j if the orbit is stable and isolated. If the orbit is stable and non-isolated, the amplitudes follow a power law.

We study the function [9, 10],

$$g(L)=\sum_n \cos(E_n^{3/4}L)\exp(-E_n^{3/2}\beta) \qquad L>0 \tag{2}$$

where $\{E_n\}$ are the energy eigenvalues of the quantum system and β is a damping factor. Using (1) one can write $g(L)$ as

$$\begin{aligned} g(L)=&\int_0^\infty \mathrm{d}E\langle d(E)\rangle\cos(E^{3/4}L)\exp(-E^{3/2}\beta)\\ &+\int_0^\infty d_{\mathrm{osc}}(E)\cos(E^{3/4}L)\exp(-E^{3/2}\beta)\,\mathrm{d}E. \end{aligned} \tag{3}$$

For the x^2y^2 potential, the average integrated density of states, $\langle N(E)\rangle\sim E^{3/2}\ln(E)$ in the asymptotic case. The first integral (I_1) can thus be carried out and yields a term which has a positive peak at $L=0$ and thereafter decays rapidly with L. It therefore contains information about the zero length orbits and hence will not be of any interest in the discussion that follows.

The second integral (I_2) can be carried out if the energy dependence of the amplitude is known. For the system under consideration, $A_{\gamma,j}\sim E^{-3/8}/\sqrt{jf(l_\gamma)}$ for periodic orbits occuring in one-parameter families. For orbits that are isolated, $A_{\gamma,j}\sim E^{-1/4}/\sin(ju_\gamma)$, $E^{-1/4}/\sinh(ju_\gamma)$ and $E^{-1/4}/\cosh(ju_\gamma)$ for stable, hyperbolic and inverse hyperbolic cases respectively. Here u_γ is the stability angle. Moreover the phase $\alpha_\gamma=\pi/4+jk_\gamma\pi/2$ for a one-parameter family and $\alpha_\gamma=jk_\gamma\pi/2$ for the isolated cases. k_γ denotes the number of conjugate points on the trajectory.

Thus, assuming the presence of a one-parameter family of periodic orbits, the integral I_2 has a term of the form,

$$J_2^\gamma = (C_\gamma/\sqrt{jf(l_\gamma)}) \int_0^\infty E^{-3/8} \cos(E^{3/4}L) \exp(-E^{3/2}\beta) \cos(E^{3/4}f(l_\gamma) - n_\gamma\pi/4)\, \mathrm{d}E$$

where C_γ is a constant depending on the measure of the one-parameter family in phase space and its degeneracy and $n_\gamma = (2k_\gamma j+1)$. Thus

$$J_2^\gamma = (C_\gamma/\sqrt{jf(l_\gamma)})[\cos(n_\gamma\pi/4)\Gamma(\tfrac{5}{12})\,{}_1F_1(\tfrac{1}{12},\tfrac{1}{2};\, b_\gamma^2/4\beta)/\beta^{5/12}$$
$$+\sin(n_\gamma\pi/4)b_\gamma\Gamma(\tfrac{11}{12})\,{}_1F_1(\tfrac{7}{12},\tfrac{3}{2};\, b_\gamma^2/4\beta)/\beta^{11/12}]\exp(-b_\gamma^2/4\beta)+T$$

where $b_\gamma = jf(l_\gamma) - L$ and T is an identical term with b_γ replaced by $a_\gamma = jf(l_\gamma)+L$. The term T therefore is not of any interest since it decays rapidly with L. The first term contributes an interesting structure to $g(L)$ as we shall now briefly describe.

For $n_\gamma\pi/2$ lying in the first quadrant, $g(L)$ increases with L in the left neighbourhood of $jf(l_\gamma)$ and attains a positive peak. It then decreases, crosses the point $jf(l_\gamma)$ and takes negative values till another orbit takes over. In the event of a periodic orbit lying in the immediate neighbourhood, interference effects would occur. A similar situation occurs for $n_\gamma\pi/2$ in the third quadrant except that the peak is now negative. For $n_\gamma\pi/2$ lying in the second quadrant, the peak is again negative but the point $jf(l_\gamma)$ now lies on the left neighbourhood of the peak (negative slope). Similarly for the fourth quadrant, the peak is positive and $jf(l_\gamma)$ lies on the left neighbourhood of the peak.

The heights of the peaks moreover depend inversely on $\sqrt{j}$, where j is the repetition number. However, a comparison should be made only between peaks for which the phase differs by integer multiples of π.

For orbits that are isolated, the behaviour is in sharp contrast to the earlier case. At the point $jf(l_\gamma)$, either a peak or a zero occurs depending on the number of conjugate points. The amplitude oscillates with j in the stable case and decays exponentially in the unstable case.

For the one-parameter family under consideration, the number of conjugate points is two. For successive repetitions, the phase therefore differs by π. Peaks should then alternate in sign and the first peak should be negative.

We have computed $g(L)$ using the first 300 energy eigenvalues and with β fixed at 0.05. The contribution of the last term in the sum of (2) is thus $\sim 10^{-4}$. The neglect of the terms due to the remaining eigenvalues has practically no effect on $g(L)$. The Schrödinger equation was solved in a suitable harmonic oscillator basis. A total of 1485 basis states were used and the matrix diagonalized using standard library routines. Convergence, however, is somewhat slow, possibly due to the 'escape channels' along the axes as discussed by Eckhardt *et al* [11]. We have compared our results with eigenvalues obtained using 1375 basis states. For the 100th eigenvalue the change is 0.07% while for the 300th one, the change is about 0.7%. Due to the (large) value of β chosen, however, the errors are considerably damped.

We provide plots for a typical positive and negative peak. Our results in the neighbourhood of $L = 34.8$ are displayed in figure 1(a). The vertical dashed line corresponds to $jf(l_\gamma)$ with $j = 1$. The characteristics mentioned above are clearly visible. The peak is negative and $f(l_\gamma)$ lies to the right neighbourhood of the peak, corresponding to $n_\gamma\pi/2$ in the third quadrant. The effect of a nearby orbit is also visible. Figure 1(b) is a similar plot around $L = 69.6$. The peak is now positive and the point $L = 69.64$ lies on the right neighbourhood indicating a phase change of π.

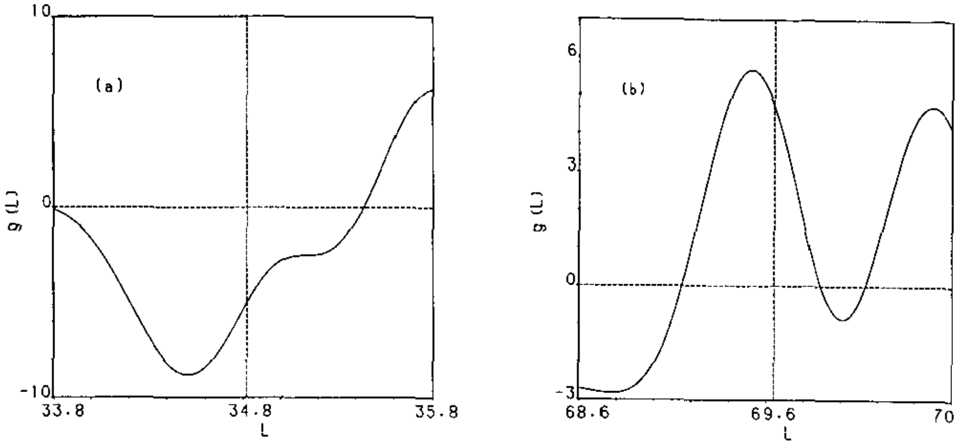

Figure 1. Plots of $g(L)$ in the neighbourhood of (a) $L=34.82$ and (b) $L=69.64$. The dashed vertical lines indicate the position of the first and second repetition of the stable periodic orbit family.

The above observations thus clearly indicate that the periodic orbit of scaled action $f(l_\gamma)=34.82$ is stable and occurs in a one-parameter family. An additional verification comes from a comparison of the amplitudes. Our results for the first six and the ninth repetitions are shown in table 1. Clearly, the signs of the third and ninth amplitudes are just the opposite of what is expected. A careful analysis of these amplitudes and their signs indicate the presence of another one-parameter family of periodic orbits with scaled action equal to $1.5f(l_\gamma)$, where $f(l_\gamma)=34.82$. Indeed, we have found this to be the case. The heights of successive peaks of this new orbit are given in table 2.

For the orbit of scaled action, 34.82, we shall thus compare the second, fourth and fifth repetitions with the first. The observed values lie within 6.45%, 6.02% and 4.08% of the expected values shown in column 3 of table 1. The differences through small, are due to a (varying) background provided by the neighbouring orbits. The power law behaviour ($j^{-1/2}$), however, is quite evident.

Table 2 shows the heights of successive peaks (B_j) for the orbit of scaled action, 52.23. The second, fourth and sixth repetitions of this orbit interfere with the third, sixth and ninth repetition of the orbit at $L=34.82$ and hence the amplitudes B_2, B_4 and B_6 cannot be 'read off' directly. Column 3 of table 2 provides the actual amplitudes of B_2 and B_6. Since both B_4 and $|A_1|\cos(6\pi)/\sqrt{6}$ have the same sign, the peak at

Table 1. The peaks A_j for successive repetitions of the periodic orbit at $L=34.82$. The third and sixth differ in sign from the expected value in column 3. The sixth is considerably enhanced. For details see text.

j	A_j	$\|A_1\|\cos(j\pi)/\sqrt{j}$
1	−8.83	−8.83
2	+5.68	+6.24
3	+5.507	−5.03
4	+4.149	+4.415
5	−4.101	−3.94
6	+4.92	+3.60
9	+3.23	−2.94

Table 2. The peaks B_j for the orbit at $L = 52.23$. The second, fourth and sixth peaks interfere with the third, sixth and ninth peaks of the previous orbit. The * in column 3 indicates an anomaly. For details see text.

j	B_j	$-\|A_1\| \cos(3j/2\pi)/\sqrt{j}$ $+\|B_j\| \cos(\pi j)$	$\|B_1\| \cos(j\pi)/\sqrt{j}$
1	−12.113	—	−12.113
2	+5.507	+10.54	+8.57
3	−7.35	—	−7.0
4	+4.92	*	+6.05
5	−6.28	—	−5.42
6	+3.23	+6.17	+4.95

$L = 208.92$ should have been considerably enhanced. However, the observed value (A_6 or B_4) is comparatively low. We do not know how to explain this anomalous behaviour. However, a comparison with the expected values given in column 4 clearly shows a power law behaviour. Moreover the magnitudes of the peaks, A_1 and B_1 of the primitive orbits at $L = 34.82$ and 52.23 indicate that the area occupied (on the surface of section) is much larger in the latter case.

We have thus clearly been able to identify the stable one-parameter family of periodic orbits of scaled action 34.82 using only the quantum spectra. We have also indicate the presence of another stable family of periodic orbits of scaled action 52.23.

The fact that it is possible to identify stable periodic orbits occupying a very small area on the surface of section (0.005% for the shorter orbit) provides hope that this method can be used to rule out ergodicity in a large class of Hamiltonian systems.

References

[1] Henon M 1983 *Chaotic Behaviour in Deterministic Systems* ed G Iooss, R H G Helleman and R Stora (Amsterdam: North-Holland)
[2] Sinai Y G 1970 *Russ. Math. Surveys* **25** 137
[3] Bunimovich L A 1974 *Funct. Anal. Appl.* **19** 254
[4] Hadamard J 1988 *J. Math. Pure Appl.* **4** 27
[5] Savvidy G K 1984 *Nucl. Phys.* B **246** 302
[6] Carnegie A and Percival I C 1984 *J. Phys. A: Math. Gen.* **17** 801
[7] Dahlqvist P and Russberg G 1990 *Phys. Rev. Lett.* **65** 2837
[8] Berry M V 1983 *Chaotic Behaviour in Deterministic Systems* ed G Iooss, R H G Helleman and R Stora (Amsterdam: North-Holland)
[9] Balazs N L and Voros A 1986 *Phys. Rep.* **143** 109
Aurich R and Steiner F 1988 *Physica* **32D** 451; 1989 *Physica* **39D** 169
Seiber M and Steiner F 1990 *Preprint* DESY 90-056
[10] Biswas D, Azam M and Lawande S V 1991 *Phys. Rev.* A **44** 4911
[11] Eckhardt B, Hose G and Pollack E 1989 *Phys. Rev.* A **39** 3776